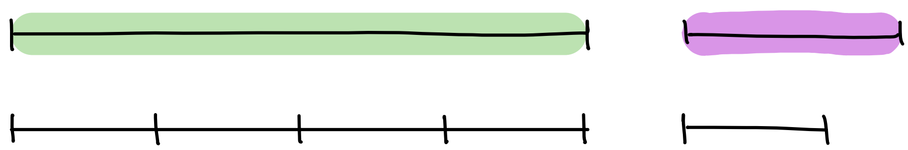
Numbers & Geometry

Numbers: For counting 1, 2, 3, 4, ...

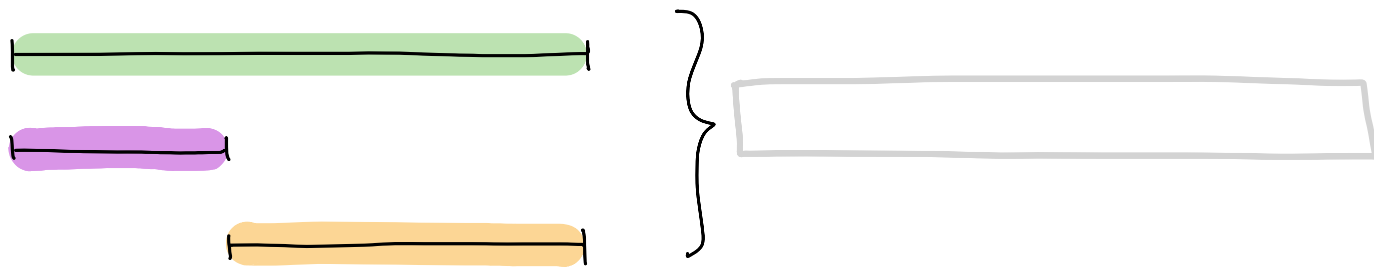
Geometry: For measuring, lengths, areas, etc.

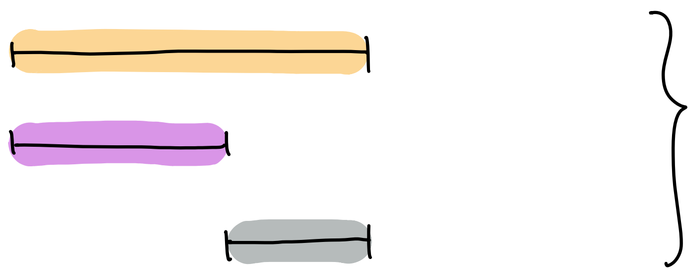
Together?



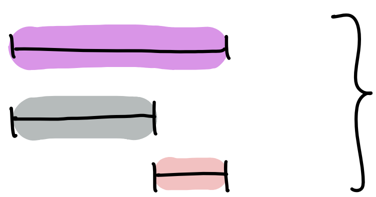
Four units long. One unit & then some.

Create a new unit?

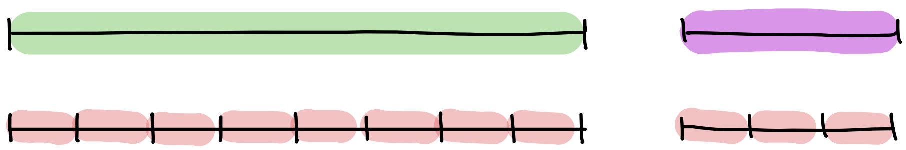




Repeat.



Repeat



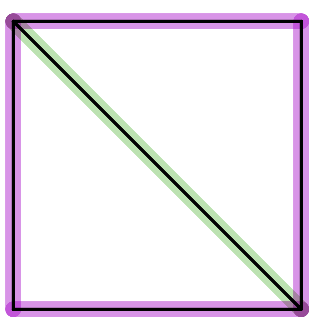
Eight units. Three units.

Another way to say this is that we have found a _______ between the two lengths; the ratio is _______________, or ____.

Question: Given two lengths is it always possible to find a common unit of measurement (so that both lengths come out even)?

... Is it always possible to find a ratio between two lengths?

Consider the _____ & the _________ of a square.

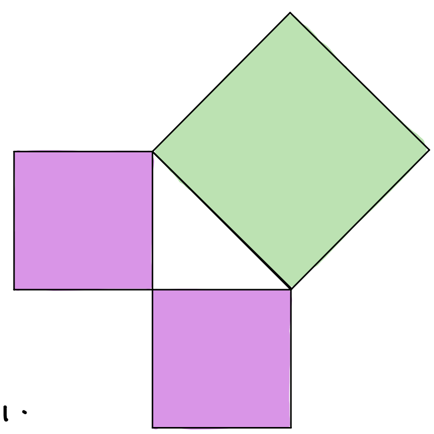


Suppose we found a common unit &

- side length : a units
- diagonal length : b units

By the Pythagorean Theorem,

$$a^2 + a^2 = b^2$$
$$2a^2 = b^2$$



This means b^2 is even & so is b, so $b = 2b_1$, for some whole # b_1.

Substitute in:

$$2a^2 = b^2 \Rightarrow 2a^2 = (2b_1)^2 = (2b_1)(2b_1) = 4b_1^2$$

$$\Rightarrow 2a^2 = 4b_1^2$$

Divide both sides by 2:

$$a^2 = 2b_1^2$$

This means a^2 is even & so is a, so $a = 2a_1$.

Substitute in as before:

$$(2a_1)^2 = 2b_1^2 \Rightarrow 4a_1^2 = 2b_1^2$$

Divide both sides by 2:

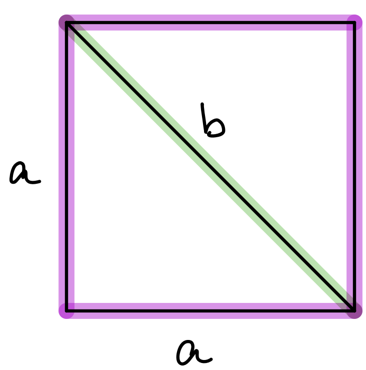
$$a_1^2 = 2b_1^2 \qquad \text{(like where we started)}$$

Continue in this way:

$$b \rightarrow b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \ldots$$

Each time, the number gets ________; can't do this forever with whole numbers.

We have shown that if we could find a whole number ratio between a & b, then we could find an infinite sequence of smaller & smaller whole numbers. Since such a sequence cannot exist, neither can the whole number ratio between a & b.

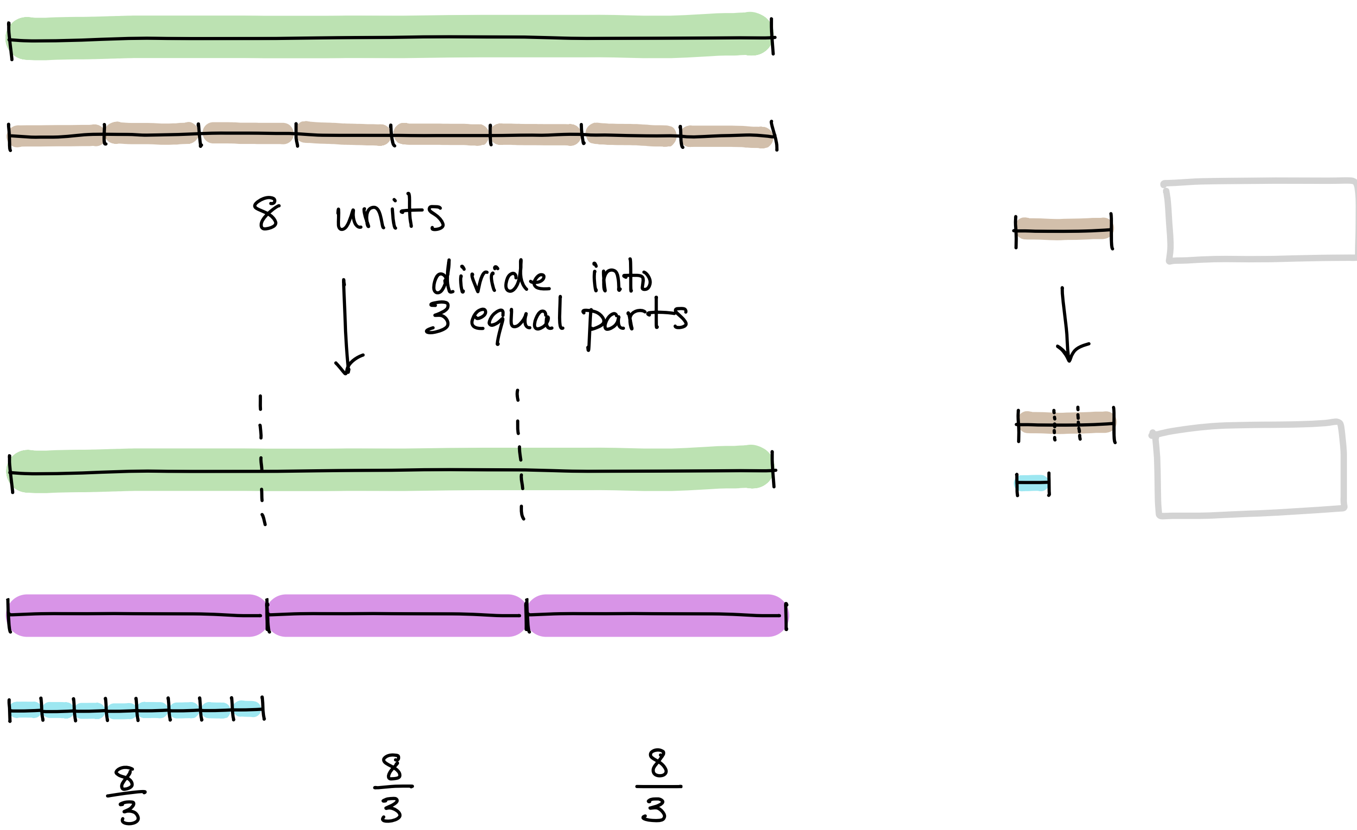


This was a crisis for the Pythagoreans & led to a centuries long rift b/w numbers & geometry.

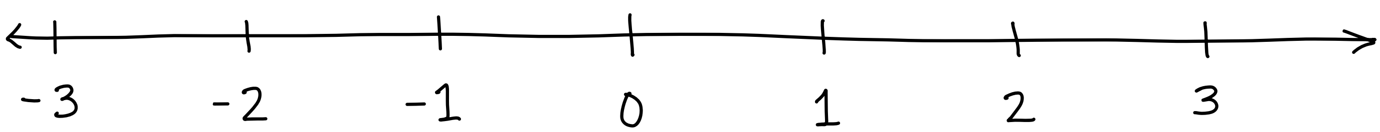
Only reunited after the invention of algebra, with Decartes' coordinate plane.

Rational and Irrational Numbers

From a modern viewpoint, ratios of whole numbers are considered numbers themselves : ____________________.

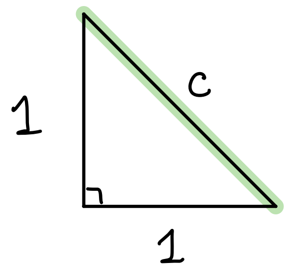


We can subdivide more & more & more... there is ________________________ rational number.



Every segment of the number line, no matter how small, contains ___________________ rational numbers.

However, there are some lengths that are not ________.



$1^2 + 1^2 = c^2 \Rightarrow 2 = c^2$

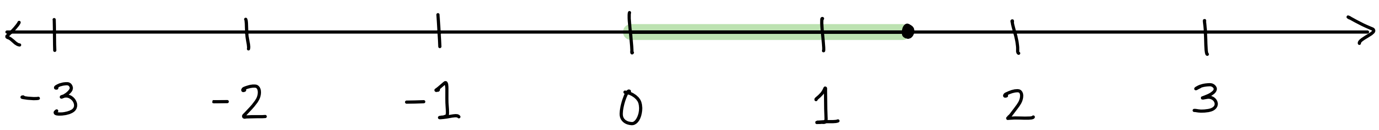
If $c = b/a$ (whole numbers)

then $2 = (b/a)^2 \Rightarrow 2a^2 = b^2 \ldots$

no such a & b.

Should the length c be considered a number?

We can put it on the number line.



In modern notation we write ________. It is considered an ________________.

Decimal Notation

Digits 1, 2, 3, 4, 5, 6, 7, 8, 9 for whole numbers less than ten

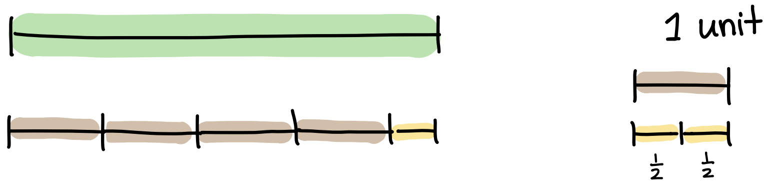
For larger whole numbers, "location, location, location"

- break up large number according to powers of ten: tens, hundreds, thousands, etc.,
- ex: 6 thousands, 2 hundreds, and 3 units
- put these digits in an appropriate "place", using a new digit, 0, as a place holder
- ex: 6203

Try a similar approach for numbers that are not whole numbers.

- split up into a whole number and a "fractional" part

Ex



4 and $1/2$ units

Write $1/2$ in terms of tenths, hundredths, thousandths, etc.

(Note: $1/10 = 10^{-1}$, $1/100 = 10^{-2}$, $1/1000 = 10^{-3}$)

Simply multiply top & bottom by 5:

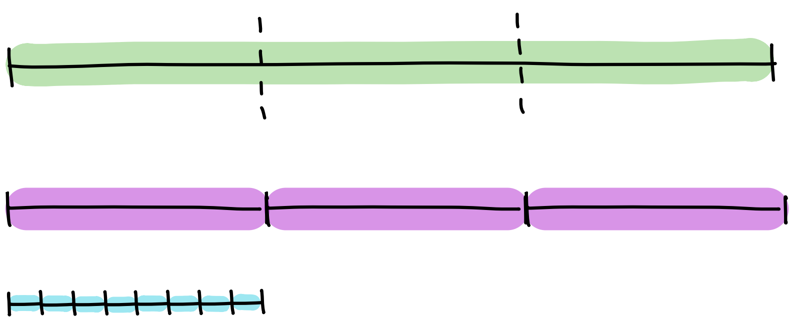
$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{5}{5} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10}$$

So $1/2$ is __________, and

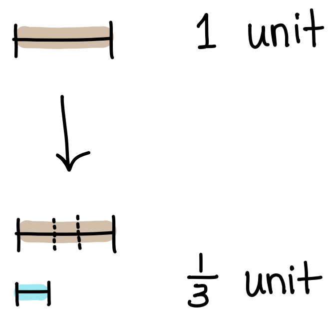
4 and $1/2$ is 4 and __ tenths.

In decimal notation: ______

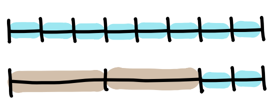
Ex $8/3$: One third of 8 units:



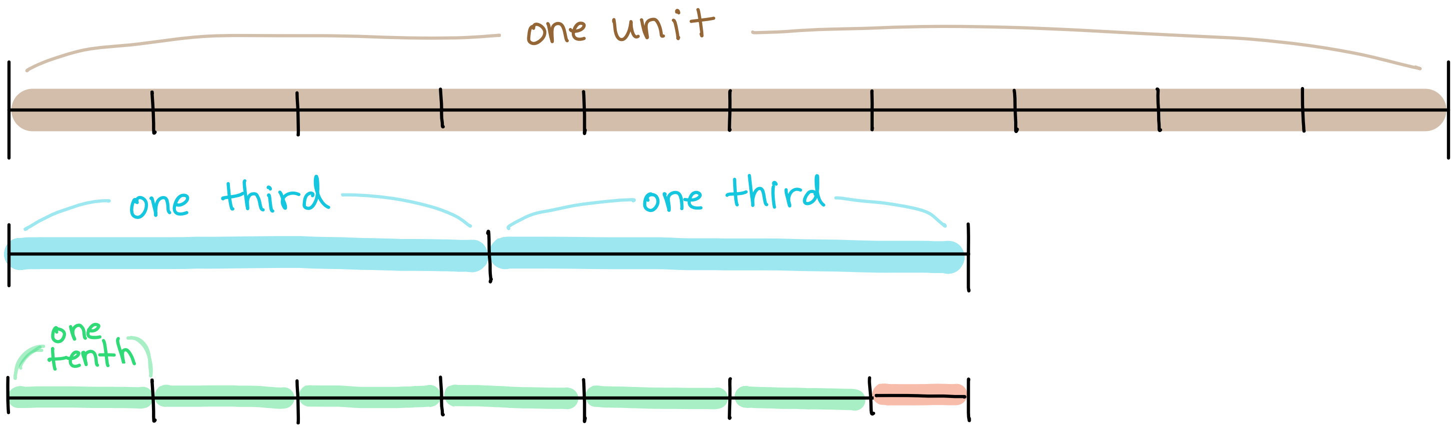
$\frac{8}{3}$ $\frac{8}{3}$ $\frac{8}{3}$



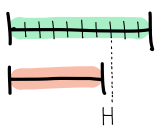
Notice: $8/3 = 2 + 2/3$



Write $2/3$ in terms of tenths, hundredths, thousandths, etc.?



$$\frac{2}{3} = \frac{6}{10} + (\text{something less than } \square)$$



$$\frac{6}{100} + (\text{something less than } \square)$$

$$\vdots$$

Try long division:

$$3\overline{)2.000000000}$$

$$\frac{2}{3} = \frac{\square}{10} + \frac{\square}{100} + \frac{\square}{1000} + \dots$$

?

What does it mean to add up infinitely many numbers?!

Write $\frac{2}{3} = \square$ (repeating decimal)

Similarly, for $1/3$

$$3\overline{)1.0000}$$

$$\frac{1}{3} = \frac{\square}{10} + \frac{\square}{100} + \frac{\square}{1000} + \dots$$

$$\frac{1}{3} = \square$$

$$\left(\frac{2}{3} = 2 \times 0.\overline{3} = 0.\overline{6} \ \checkmark\right)$$

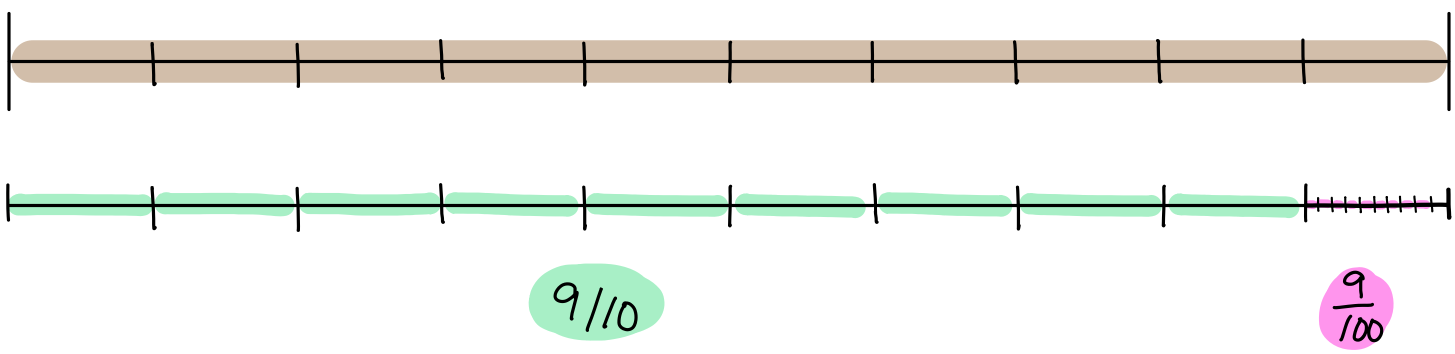
Fact Given a rational number p/q in lowest terms,

- if q is divisible by ___ and/or ___ but no other primes, p/q can be written as a ________ decimal
  ex: $1/2 = 0.5$, $3/5 = 0.6$, $9/20 = 0.45$, etc. ;
- otherwise p/q can be written as a ________ decimal
  ex: $8/3 = 2.\overline{6}$, $1/7 = 0.\overline{142857}$, $8/21 = \overline{0.380952}$, etc..

This is NOT obvious! Quite sophisticated! (1800s)

Fact $0.\overline{9} = 1$

$$0.\overline{9} = \square/10 + \square/100 + \square/1000 + \square/10000 + \dots$$



Notice that:

$$1 - 0.9 = 0.1 = \square$$
$$1 - 0.99 = 0.01 = \square$$
$$1 - 0.999 = 0.001 = \square$$
$$1 - 0.999 = 0.0001 = \square$$
$$\vdots$$

If $0.\overline{9} < 1$, then there must be a ________ between the two, but the ________ would have to be smaller than all negative powers of ten: 10^{-n} for all n. ________